Phi: That Golden Number

by Mark Freitag

Most people are familiar with the number Pi, since it is one of the most ubiquitous irrational numbers known to man. But, there is another irrational number that has the same propensity for popping up and is not as well known as Pi. This wonderful number is Phi, and it has a tendency to turn up in a great number of places, a few of which will be discussed in this essay.

One way to find Phi is to consider the solutions to the equation

$$x^2 - x - 1 = 0$$

When solving this equation we find that the roots are

$$\mathbf{x = \frac{1+\sqrt{5}}{2} \sim 1.618... \text{ or } x= \frac{1\ \sqrt{5}}{2} \sim -.618...}$$

We consider the first root to be Phi. We can also express Phi by the following two series.

$$\textbf{Phi} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+...}}}}} \quad \textbf{or Phi} = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1...}}}}}$$

We can use a spreadsheet to see that these two series do approximate the value of Phi.

Number of Iterations	Approximation of Phi by Roots	Approximation of Phi by Fractions
1	1	2
2	1.414213562	1.5
3	1.553773974	1.666666667
4	1.598053182	1.6
5	1.611847754	1.625
6	1.616121207	1.615384615
7	1.617442799	1.619047619
8	1.617851291	1.617647059
9	1.617977531	1.618181818
10	1.618016542	1.617977528
11	1.618028597	1.618055556
12	1.618032323	1.618025751
13	1.618033474	1.618037135
14	1.61803383	1.618032787
15	1.61803394	1.618034448
16	1.618033974	1.618033813
17	1.618033984	1.618034056
18	1.618033987	1.618033963
19	1.618033988	1.618033999

Or, we can show that the limit of the infinite series equals Phi in a more concrete way. For example, let x be equal to the infinte series of square roots.

$$x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1\ldots}}}}}\,.$$

Squaring both sides we have

$$x^2 - 1 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1\ldots}}}}}$$

But this leads to the equation

$$x^2 - 1 = x$$

which in turn leads to

$$x^2 - x - 1 = 0$$

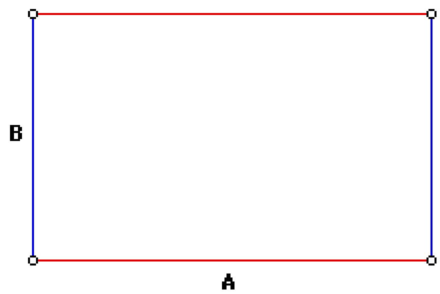
and this has Phi as one of its roots. Similarly, it can be shown that the limit of the series with fractions is Phi as well. When finding the limit of the fractional series, we can take a side trip and see that Phi is the only number that when one is subtracted from it results in the reciprocal of the number.

Phi can also be found in many geometrical shapes, but instead of representing it as an irrational number, we can express it in the following way. Given a line segment, we can divide it into two segments A and B, in such a way that the length of the entire segment is to the length of the segment A as the length of segment A is to the length of segment B. If we calculate these ratios, we see that we get an approximation of the Golden Ratio.

$$\frac{A+B}{A} = 1.618 \qquad \frac{A}{B} = 1.618$$

I have creted a GSP script for dividing a segment (given its endpoints) into the Golden ratio. If you would like to explore this, click here.

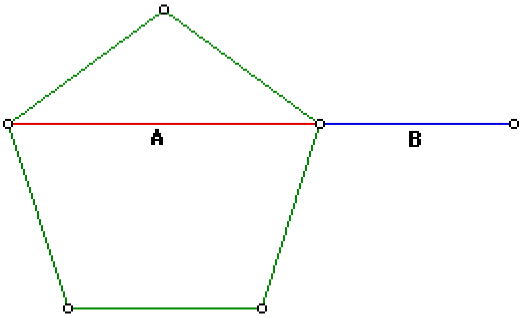
Another geometrical figure that is commonly associated with Phi is the Golden Rectangle. This particular rectangle has sides A and B that are in proportion to the Golden Ratio. It has been said that the Golden Rectangle is the most pleasing rectangle to the eye. If fact, it is said that any geometrical shape that has the Golden Ratio in it is the most pleasing to look at of those types of figures. Anyhow, here is a picture of the Golden Rectangle.



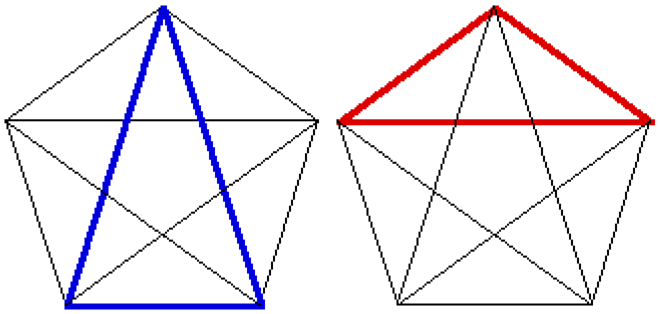
$$\frac{A}{B} = 1.618$$

Keep this rectangle in mind, I'll be coming back to it later. But right now, I want to show where The Golden Ratio (Phi) pops up in other geometrical figures.

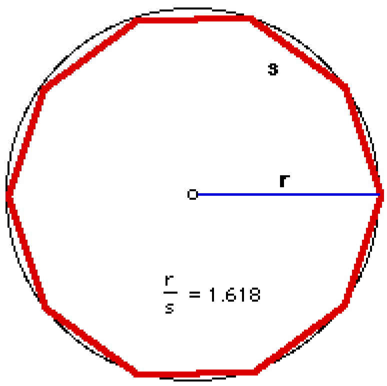
We can use the Golden Section to construct a regular pentagon.



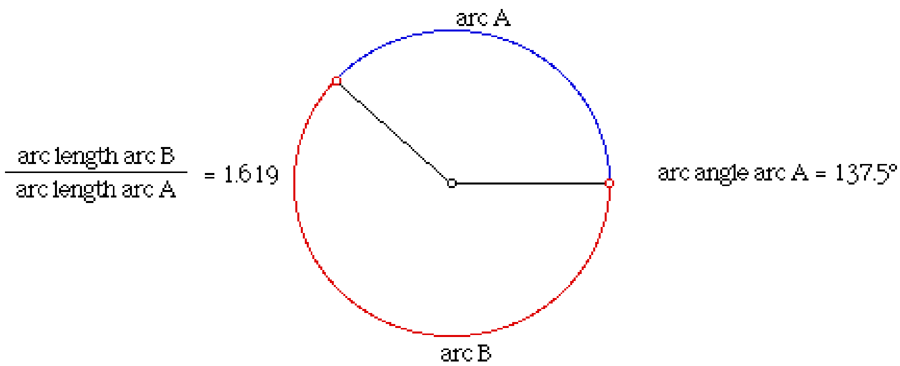
If you would like to see a GSP script of this construction, click here. If we connect the vertices of the regular pentagon, we can get two different Golden Traingles. The blue triangle has its sides in the golden ratio with its base, and the red triangle has its base in the golden ratio with one of the sides.



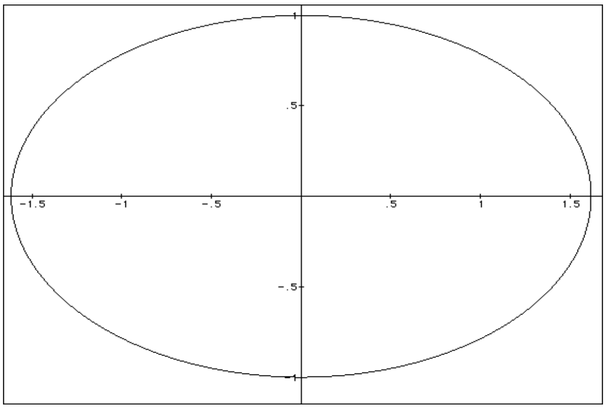
If we inscribe a regular decagon in a circle, the ratio of a side of the decagon to the radius of the circle forms the golden ratio.



If we divide a circle into two arcs in the proportion of the Golden ratio, the central angle of the smaller arc marks off the Golden Angle, is 137.5 degrees.



We can also form a Golden Ellipse. This ellipse has its two axes in the Golden Ratio.



Let's turn back to one of the Golden Triangles for a moment. If we take the isoceles triangle that has the two base angles of 72 degrees and we bisect one of the base angles, we should see that we get another Golden triangle that is similar to the first (Figure 1). If we continue in this fashion we should get a set of Whirling Triangles (Figure 2).

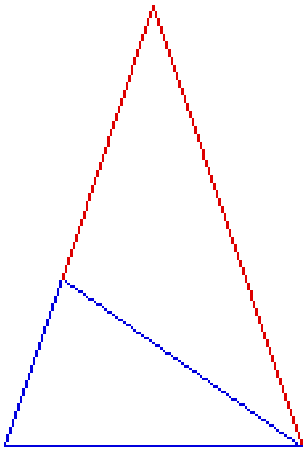
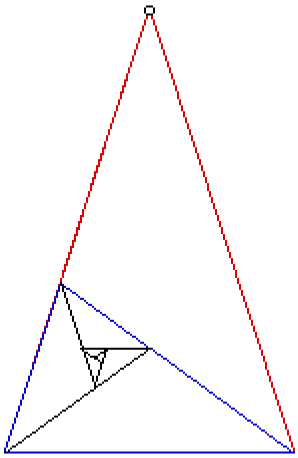



Figure 1 Figure 2

Out of these Whirling Triangles, we are able to draw a logarithmic spiral that will converge at the intersection of the the two blue lines in Figure 3.

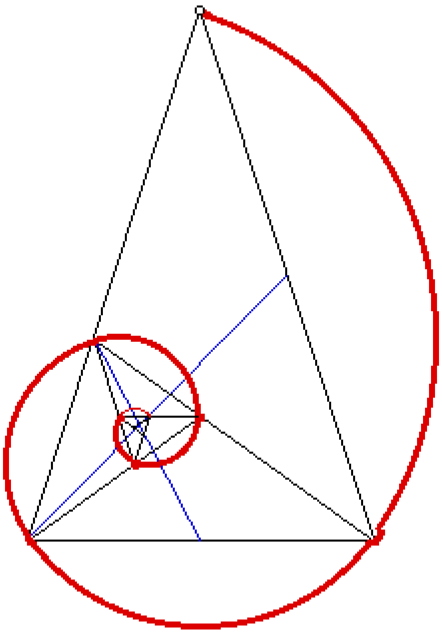
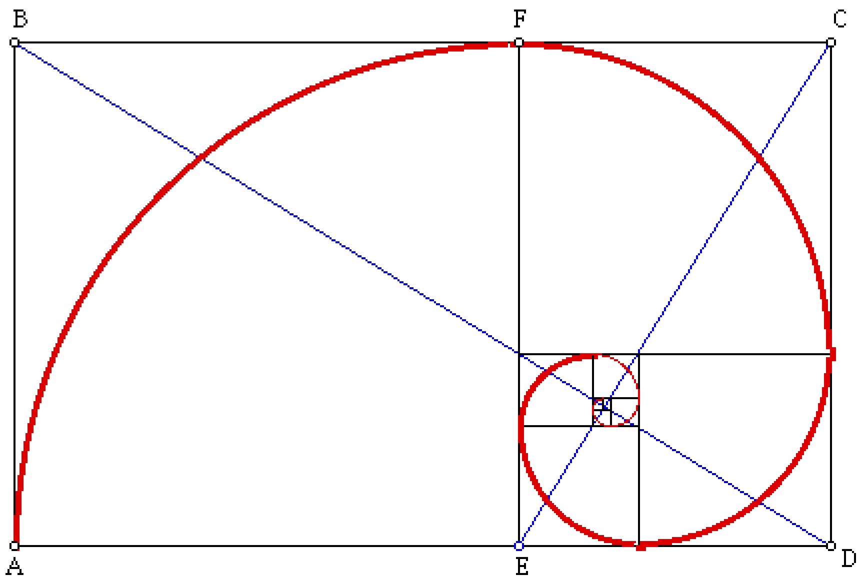


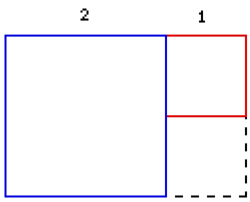
Figure 3

We can do a similar thing with the Golden Rectangle. We can make a set of Whirling Rectangles that produces a similar logarithmic spiral. Again this spiral converges at the intersection of the two blue lines, and these ratio of the lengths of these two lines is in the Golden Ratio.

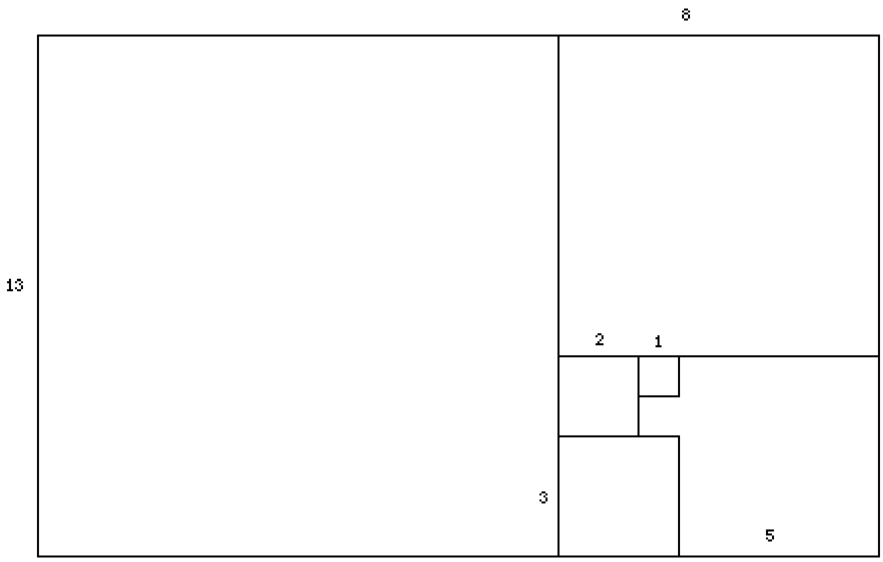


I will outline the proof that the ratio of the lengths of the two diagonals is indeed the Golden ratio. Assume that rectangle ABCD is a Golden Rectangle. Hence, AD/AB =AE/ED. But, FE = AE, and so FE/ED= Phi. Hence, rectangle FCDE is a Golden Rectangle. We have two similar rectangles and so since Phi = AD/EF then BD/CE = Phi.

An interesting thing happens when we work with these whirling rectangle. Suppose we take a rectangle of side 1 unit and a rectangle of side 2 units and we put them side to side in the following fashion and draw the appropriate segments to form a rectangle.



If we continue to create rectangles in this fashion we will get a series of whirling rectangles like the one of above. The following picture shows several such rectangles, and the lengths of their sides.



If we take ratios of the length we will see that the series of whirling rectangles will begin to estimate the Golden Ratio.

2/1 = 2 3/2= 1.5 5/3 = 1.666... 8/5 = 1.6 13/8 = 1.625 and so on.

Hence as we increase the number of squares we get a figure that begins to look more and more like the Golden Rectangle. It might also be noticed that there is something special about the sides of the squares. If we list them we have, 1, 2, 3, 5, 8, 13, ... This of course is the famous Fibonacci sequence. As will be shown in the rest of the essay, the Fibonacci sequence and the golden ratio are intertwined with each other.

Leonardo Fibonacci began study of this sequence by posing the following problem in his book, *Liber Abaci*

How many pairs of rabbits will be produced in a year, beginning with a single pair , if in every month each pair bears a new pair which becomes productive from the second month on?

Of course, this problem gives rise to the sequence 1, 1, 2, 3, 5, 8, 13, ... in which any term after the first two can be found by summing the two previous terms. In functional notation we could write $f(n) = f(n-1) + f(n-2)$ using $f(0) = 1$ and $f(1) = 1$.This particular sequence has some interesting properties. Instead of proving these, I will justify them by use of a spreadsheet.

I began by finding the first 40 Fibonacci numbers, which will be used in most of the demonstrations. The values can be found in column A. One of the first propeties that might be noticed is that for two positive integers a and b, if $a \mid b$ then $F(a) \mid F(b)$. For example, if $a = 5$ and
$b = 10$, then we have $F(a) = 5$ and $F(b) = 55$. The converse also holds true (If $F(a) \mid F(b)$, then $a \mid b$), and proof of these two properties can be found in Maxfield and Maxfield (1972).

Another property of the Fibonnaci numbers is that no two consecutive numbers in the sequence have a common prime factor. As an example consider F(7) through F(10) (All though we can look at any of the first 40 numbers), and see that no consectutive pair has a common factor.

$$F(7) = 13 = 1 \times 13$$
$$F(8) = 21 = 3 \times 7$$
$$F(9) = 34 = 2 \times 17$$
$$F(10) = 55 = 5 \times 11$$

The proof is as follows.

Let $F(n+1)$ and $F(n)$ be two consecutive Fibonnaci numbers and suppose that $p \mid F(n+1)$ and $p \mid F(n)$. Since $F(n+1) = F(n) + F(n-1)$, then $p \mid F(n) + F(n-1)$ and so $p \mid F(n-1)$. We also know that $F(n) = F(n-1) + F(n-2)$, and by similar argument we get that $p \mid F(n-2)$. If we continue in this manner, we'll see that p divides every Fibonnaci number less than $F(n+1)$. Hence $p \mid F(2)$ and $p \mid F(3)$. Since $F(2) = 2$ and $F(3) = 3$, then p must equal 1. Therefore, 1 is the only number that will divide both $F(n+1)$ and $F(n)$ and we conclude that they are relatively prime.

Another property that is verifiable by looking at the first forty terms of the sequence, is that the Fibonnaci sequence is complete with respect to the positive integers. This means that every positive integer can be written as the sum of finite terms from the sequence with no term being used more than once. For example, consider the positive integer 257.

$$257 = 233 + 21 + 3 = F(13) + F(8) + F(4)$$

This representation is not unique since

$$257 = 233 + 13 + 8 + 2 + 1 = F(13) + F(7) + F(6) + F(3) + F(2)$$

The proof of this property is a straight forward inductive proof and can be found in Maxfield and Maxfield (1972).

As was seen with the whirling rectangles, the ratio of consectutive Fibonacci numbers begins to apporximate the Golden Ratio. A spreadsheet can be used to see that this is more clearly the case. Values of these ratios can be found in column B.

The proof that the $\lim_{n \to \infty} F_{n+1}/F_n$ is as follows. For the rest of this discussion we will let $\beta = \frac{1+\sqrt{5}}{2} = 1.61803$, and $\beta' = \frac{1-\sqrt{5}}{2} = -.61803$.

To begin to show this limit, it is important to realize that each Fibonnaci number $F_n = \beta^n - (\beta')^n / \beta - \beta'$ (Binet's Formula). Thus the

$$\lim_{n\to\infty} F_{n+1} / F_n = \lim_{n\to\infty} \frac{\frac{\beta^{n+1} - (\beta')^{n+1}}{\beta - \beta'}}{\frac{\beta^n - (\beta')^n}{\beta - \beta'}} = \lim_{n\to\infty} \frac{\beta^{n+1} - (\beta')^{n+1}}{\beta^n - (\beta')^n} =$$

$$= \beta \lim_{n\to\infty} \frac{\beta^n}{\beta^n - (\beta')^n} - \beta' \lim_{n\to\infty} \frac{(\beta')^n}{\beta^n - (\beta')^n}$$

$$= \beta \lim_{n\to\infty} \frac{1}{\frac{\beta^n}{\beta^n} - \frac{(\beta')^n}{\beta^n}} - \beta' \lim_{n\to\infty} \frac{1}{\frac{\beta^n}{(\beta')^n} - \frac{(\beta')^n}{(\beta')^n}} = \beta \lim_{n\to\infty} 1 - 0 = \beta = \frac{1+\sqrt{5}}{2}.$$

The second ratio that was considered was the ratio of every other term, or F(n + 2)/ F(n). The values for the computed ratios for n = 1 to 40 are found in column C. As n increased, it could be seen that there was again a limiting value of approximately 2.61803. This number also has significance with regard to the golden ratio. We know that ß is one solution to the equation x^2 = x + 1, and this equation tells us that ß^2 = ß + 1. When we do the arithmetic, we find that ß^2 = 2.61803. The conclusion that we can then make from this is that the ratio of F(n + 2)/ F(n) is an estimate of ß^2, and this estimate gets better as n gets larger.

Other ratios were of interest as well and the third ratio to be considered was the ratio of every third term, or F(n + 3)/ F(n). The values for the computed ratios for n = 1 to 40 are found in column D. As n increased, it could be seen that there was again a limiting value of approximately 4.23607. This number also has significance with regard to the golden ratio. If we consider the system of equations

x^3 = x^2 + x
x^2 = x + 1

and make a substitution, we find that x^3 = 2x + 1. This means that if ß was a solution to the equation x^2 = x + 1, then we should be able to conclude that ß^3 = 2ß + 1. So going through the arithmetic, we find that 2ß + 1= 2.61803. So the next connection that we make is that the ratio of
F(n + 2) / F(n) has a limiting value of ß^3.

Finally the fourth ratio to be considered was the ratio of every fourth term, or F(n + 4) / F(n). The values for the computed ratios for n = 1 to 40 are found in column E of the spreadsheet. As n increased, it could be seen that there was a limiting value of approximately 6.8541, and again this number has connections with the golden ratio. If we consider the system of equations

x^4 = x^3 + x^2
x^3 = x^2 + x
x^2 = x + 1

and make appropriate substitutions, we find that x^4 = 3x + 2. This means that if ß was a solution to the equation x^2 = x + 1, then we should be able to conclude that ß^4 = 3ß + 2. So going through the arithmetic, we find that 3ß + 2 = 6.8541. So the final connection we make is that the ratio of
F(n + 4) / F(n) has a limiting value of ß^4.

So what hypotheses can we make from this sequence of discussions? First, if we were to continue to solve equations, we could find a general formula for any power of x. This being

x^(n + 1) = x^n + x^(n - 1) = F(n - 1) x + F(n - 2).

Using this equation, we can compute any power of ß simply substituting ß in for x. The proofs of these facts should follow from a simple inductive process. The last hypothesis that we make is that, for any integer k> 1,

$$\lim_{n\to\infty} F_{n+k}/F_n = ß^k.$$

The proof is very similar to the previous one.

$$\lim_{n\to\infty} F_{n+k}/F_n = \lim_{n\to\infty} \frac{\dfrac{ß^{n+k} - (ß')^{n+k}}{ß - ß'}}{\dfrac{ß^n - (ß')^n}{ß - ß'}} = \lim_{n\to\infty} \frac{ß^{n+k} - (ß')^{n+k}}{ß^n - (ß')^n} =$$

$$= ß^k \lim_{n\to\infty} \frac{ß^n}{ß^n - (ß')^n} - (ß')^k \lim_{n\to\infty} \frac{(ß')^n}{ß^n - (ß')^n}$$

$$= ß^k \lim_{n\to\infty} \frac{1}{\dfrac{ß^n}{ß^n} - \dfrac{(ß')^n}{ß^n}} - (ß')^k \lim_{n\to\infty} \frac{1}{\dfrac{ß^n}{(ß')^n} - \dfrac{(ß')^n}{(ß')^n}} = ß^k \lim_{n\to\infty} 1 - 0 = ß^k$$

The previous discussion has dealt with only one of the solutions to the equation $x^2 = x + 1$. By simple algebra, we find that the other root of this equation is $\frac{1-\sqrt{5}}{2}$. We will call this number ß'. Do the same properties hold for this number, that held for the other?

The first question to be answered would be, "Is ß' the limiting value of some ratio of the Fibonnaci numbers as they go towards infinity? The answer is yes. If we consider the following ratio
F(n)/ F(n + 1) and consider the limit of this as n goes to infinity, we do get the value of ß' = -.61803. The value of the first thirty-nine ratios can be found in column F of the spreadsheet. We can also go through a similar process with the ratios of F(n)/F(n + 2), F(n)/F(n + 3), and F(n)/F(n +4), and see that the same properties hold for ß' that held for ß. The values for these ratios are in the spreadsheet columns G, H, and I respectively.

The spreadsheet was used in another way to help analyze the Fibonnaci sequence. It allowed us to easily change the first two numbers of the sequence and therefore allowing us to change the entire sequence. Many different values were tried including positive and negative integers and decimals of varying length. The very interesting event that happened was that the ratios always estimated their appropriate power of ß of ß'.

Also, the spreadsheet enabled us to look at the formula F(n - 1) * F(n + 1) - F(n)^2 for n a positive integer less than or equal to 40. The resulting values are in the spreadsheet column J. The values are somewhat interesting. When n is even we get 1, and when n is odd we get -1. Hence, we hypothesize that

$$F_{n-1} * F_{n+1} - F_n^2 = (-1)^n.$$

The proof is as follows.

We proceed by induction and begin with the case n = 2. Then $F_1 = 1$, $F_2 = 1$, and $F_3 = 2$. So substituting into the formula we have $1 * 2 - 1 = 1 = (-1)^2$. For n = 3, we have $1*3 - 2^2 = -1 = (-1)^3$. So we assume that the formula holds for n = k and so we have $F_{k-1} * F_{k+1} - F_k^2 = (-1)^k$. The next step is to show that it holds true for n = k + 1. From the way that the sequence is designed, we know that $F_k = F_{k+2} - F_{k+1}$ and $F_{k-1} = F_{k+1} - F_k$. We substitute these equations in the following way

$$F_{k-1} * F_{k+1} - F_k^2 = F_{k+1}(F_{k+1} - F_k) - F_k(F_{k+2} - F_{k+1}) = (F_{k+1})^2 - F_k * F_{k+2}.$$

Thus, $(F_{k+1})^2 - F_k * F_{k+2} = (-1)^k$ and multiplying both sides of the equation by -1, we get $F_k * F_{k+2} - (F_{k+1})^2 = -(-1)^k = (-1)^{k+1}$.

The investigations that deal with the Fibonnaci sequence are abundant, and there are many articles and books that deal with these. It is certain that the use of a spreadsheet could help explore these investigations.

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A	B	C	D	E	F	G	H	I	J
1									
1	1				1				
2	2	2			0.5	0.5			1
3	1.5	3	3		0.66667	0.33333	0.33333		-1
5	1.66667	2.5	5	5	0.6	0.4	0.2	0.2	1
8	1.6	2.66667	4	8	0.625	0.375	0.25	0.125	-1
13	1.625	2.6	4.33333	6.5	0.61538	0.38462	0.23077	0.15385	1
21	1.61538	2.625	4.2	7	0.61905	0.38095	0.23810	0.14286	-1
34	1.61905	2.61538	4.25	6.8	0.61765	0.38235	0.23529	0.14706	1
55	1.61765	2.61905	4.23077	6.875	0.61818	0.38182	0.23636	0.14545	-1
89	1.61818	2.61765	4.23810	6.84615	0.61798	0.38202	0.23596	0.14607	1
144	1.61798	2.61818	4.23529	6.85714	0.61806	0.38194	0.23611	0.14583	-1
233	1.61806	2.61798	4.23636	6.85294	0.61803	0.38197	0.23605	0.14592	1
377	1.61803	2.61806	4.23596	6.85455	0.61804	0.38196	0.23607	0.14589	-1
610	1.61804	2.61803	4.23611	6.85393	0.61803	0.38197	0.23607	0.14590	1
987	1.61803	2.61804	4.23605	6.85417	0.61803	0.38197	0.23607	0.14590	-1
1597	1.61803	2.61803	4.23607	6.85408	0.61803	0.38197	0.23607	0.14590	1
2584	1.61803	2.61803	4.23607	6.85411	0.61803	0.38197	0.23607	0.14590	-1
4181	1.61803	2.61803	4.23607	6.85410	0.61803	0.38197	0.23607	0.14590	1
6765	1.61803	2.61803	4.23607	6.85410	0.61803	0.38197	0.23607	0.14590	-1
10946	1.61803	2.61803	4.23607	6.85410	0.61803	0.38197	0.23607	0.14590	1
17711	1.61803	2.61803	4.23607	6.85410	0.61803	0.38197	0.23607	0.14590	-1
28657	1.61803	2.61803	4.23607	6.85410	0.61803	0.38197	0.23607	0.14590	1
46368	1.61803	2.61803	4.23607	6.85410	0.61803	0.38197	0.23607	0.14590	-1

Certainly there are many more areas to explore. For example, can the ratios in columns B through E be represented geometrically? What other ratios can I form using the Fibonacci sequence and still have the limit of the ratios converge to a particular number? What relationship would these numbers have to the Golden Ratio? I'll leave these questions for other people to answer, for right now, I think that it is time I moved on to something else.

References

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Gardner, M. (1961). The second scientific American book of mathematical puzzles and diversions. Simon and Schuster: New York, NY.